Kuwait University | Math 102 | April 10, 2011
Dept. of Mathematics | Exam 1 | Duration 90 min

Calculators and mobile phones are not allowed

Question 1. (2+2+2 pts) Let $f(x) = \sin^{-1}(2-x) + \ln(4-x^2)$.

(a) Find the domain of f.

(b) Show that f has an inverse function.

(c) Find the domain of f^{-1}.

Question 2. (2+2 pts)

(a) Solve the equation $\log_2(x^2+1) - 2\log_2 x = 1$.

(b) Show that $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$, for any $x \in (-\infty, \infty)$.

Question 3. (3 pts). Find $\dfrac{dy}{dx}$ if $y = \left(\dfrac{\pi}{2} - \tan^{-1} x\right)^{\tanh x}$.

Question 4. (3+3 pts). Evaluate the integrals:

(a) $\displaystyle\int \frac{10^{1/x} - \tan(1/x)}{x^2}\,dx$.

(b) $\displaystyle\int \frac{\sin x}{\sqrt{8+\sin^2 x}}\,dx$.

Question 5. (3 pts). Find the limit, if it exists: $\displaystyle\lim_{x\to\infty} (\cosh x)^{1/x}$.

Question 6. (1.5 + 1.5 pts). Answer the following as True or False. Justify your answer.

(a) $\log_{1/3} 10 > 0$.

(b) $\sin^{-1}(\sin 2) = 2$.

SOLUTIONS

1. a. Conditions: $-1 \le 2 - x \le 1 \iff x \in [1,3]$; $4 - x^2 > 0 \iff x \in (-2,2)$ $\Big\}$ So, the domain is $[1,2)$.

b. $f'(x) = \dfrac{-1}{\sqrt{1-(2-x)^2}} - \dfrac{2x}{4-x^2} < 0 \Rightarrow f$ is decreasing $\Rightarrow f$ is one-to-one, so f has an inverse.

c. $f(1) = \sin^{-1}(1) + \ln 3 = \frac{\pi}{2} + \ln 3$, $\lim\limits_{x \to 2^-} f(x) = -\infty$, so we have: domain (f^{-1}) = range $(f) = (-\infty, \frac{\pi}{2} + \ln 3]$.

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2. a. Condition: $x > 0$. We have: $\log_2(x^2+1) - \log_2(x^2) = \log_2 2 \iff \log_2 \dfrac{x^2+1}{x^2} = \log_2 2 \iff x^2 = 1$ (since $\log_2 u$ is 1-1). Solution: $x = 1$.

b. Put $\tan^{-1}x = y \iff x = \tan y$. Then the left side becomes $\sin y$. Similarly, the right part becomes: $\dfrac{x}{\sqrt{1+x^2}} = \dfrac{\tan y}{\sqrt{1+\tan^2 y}} = \dfrac{\tan y}{\sqrt{\sec^2 y}} = \dfrac{\tan y}{\sec y} = \sin y$, where we use $\sec y > 0$ for all $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence we proved the identity.

(it is given 1.5p for a proof for $x > 0$, by using the right triangle).

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3. Apply „ln" to both sides and take derivatives: $\ln y = \tanh x \ln\left(\frac{\pi}{2} - \tan^{-1}x\right)$

$\Rightarrow \dfrac{y'}{y} = \operatorname{sech}^2 x \ln\left(\frac{\pi}{2} - \tan^{-1}x\right) + \tanh x \dfrac{-\frac{1}{1+x^2}}{\frac{\pi}{2} - \tan^{-1}x}$, which implies

$$y' = \left(\frac{\pi}{2} - \tan^{-1}x\right)^{\tanh x} \left\{ \operatorname{sech}^2 x \ln\left(\frac{\pi}{2} - \tan^{-1}x\right) - \frac{\tanh x}{(1+x^2)\left(\frac{\pi}{2} - \tan^{-1}x\right)} \right\}.$$

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4. a. Substitution: $\frac{1}{x} = u$, $\frac{1}{x^2}dx = -du$. Then we have:

$$I = -\int (10^u - \tan u)\,du = -\int 10^u du + \int \tan u\, du = -\frac{1}{\ln 10} 10^u + \ln|\sec u| + C$$
$$= -\frac{1}{\ln 10} 10^{1/x} + \ln\left|\sec \tfrac{1}{x}\right| + C.$$

b. First, write $I = \displaystyle\int \frac{\sin x}{\sqrt{8+\sin^2 x}}\,dx = \int \frac{\sin x}{\sqrt{9-\cos^2 x}}\,dx$. Then, make the substitution $\cos x = t$ and obtain $I = -\displaystyle\int \frac{dt}{\sqrt{9-t^2}} = \cos^{-1}\left(\frac{t}{3}\right) + C = \cos^{-1}\left(\frac{\cos x}{3}\right) + C$.

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5. Check if it is indeterminate form: $\lim\limits_{x\to\infty} (\cosh x)^{1/x} = \infty^0$. Put $y = (\cosh x)^{1/x}$ and calculate: $\lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln(\cosh x)}{x} = \dfrac{\infty}{\infty}$ (I.F.). Apply L'H Rule and obtain $\lim\limits_{x\to\infty} \dfrac{\ln(\cosh x)'}{(x)'} = \lim\limits_{x\to\infty} \tanh x = 1$. So, $\lim\limits_{x\to\infty} \ln y = 1$, and thus $\lim\limits_{x\to\infty} (\cosh x)^{1/x} = e$.

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6. a. False: $\log_{1/3} x$ is decreasing, so from $10 > 1$ we deduce $\log_{\frac{1}{3}} 10 < \log_{\frac{1}{3}} 1 = 0$.

b. False: $2 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.